$$\beta(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma(m+n)}$$

We know that $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\,dx$

$$\Gamma(m) = 2\int_0^\infty e^{-y^2} y^{2m-1}\,dy$$

$$\therefore \Gamma(m)\Gamma(n) = \left[2\int_0^\infty e^{-y^2} y^{2m-1}\,dy\right]\left[2\int_0^\infty e^{-x^2} x^{2n-1}\,dx\right]$$

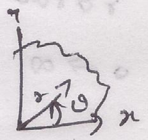
$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\,dx\,dy$$

$x : 0 \to \infty$

$y : 0 \to \infty$

We use polar coordinates to evaluate the above integral

i.e., $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$.



$r : 0 \to \infty \qquad \theta : 0 \to \pi/2$

$$\Gamma(m)\Gamma(n) = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1}\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_{r=0}^{\infty} e^{-r^2} r^{2n-1+2m-1+1} \cos^{2n-1}\theta \sin^{2m-1}\theta\,dr\,d\theta$$

$$= 2\int_0^{\infty} e^{-r^2} r^{2(m+n)-1}\,dr \times 2\int_0^{\pi/2} \cos^{2n-1}\theta \sin^{2m-1}\theta\,d\theta$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\,\beta(m,n)$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$